Designing Truthful Spectrum Double Auctions with Local Markets

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Abstract—Market-driven spectrum auctions offer an efficient way to improve spectrum utilization by transferring unused or underused spectrum from its primary license holder to spectrum-deficient secondary users. Such a spectrum market exhibits strong locality in two aspects: 1) that spectrum is a local resource and can only be traded to users within the license area, and 2) that holders can partition the entire license areas and sell any pieces in the market. We design a spectrum double auction that incorporates such locality in spectrum markets, while keeping the auction economically robust and computationally efficient. Our designs are tailored to cases with and without the knowledge of bid distributions. Complementary simulation studies show that spectrum utilization can be significantly improved when distribution information is available. Therefore, an auctioneer can start from one design without any a priori information, and then switch to the other alternative after accumulating sufficient distribution knowledge. With minor modifications, our designs are also effective for a profit-driven auctioneer aiming to maximize the auction revenue.

Index Terms—Dynamic spectrum access, spectrum double auction, local markets, truthfulness, uniform pricing, discriminatory pricing

1 INTRODUCTION

The recent explosive growth of wireless networks, with their ever-growing demand for radio spectrum, has exacerbated the problem of spectrum scarcity. Such scarcity, however, is not an outcome of exhausted physical spectrum, but a result of inefficient channel utilization due to existing policies that channels are licensed to their authorized holders (typically those who win government auctions of spectrum), and unlicensed access is not allowed even if the channel is not used.

To utilize such idle channels and to improve their utilization, it is critical to design sufficient incentives that encourage primary license holders to allow other spectrum-deficient users to access these channels. It is intuitive to observe that underused channels have values that can be efficiently determined by a market, governed by spectrum auctions. If designed well, a spectrum auction offers an efficient way to create a market: it attracts both license holders and wireless users to join, and to either buy or sell idle channels in the market. Once a transaction is conducted, the seller (license holder) earns extra income by leasing unused channels to the buyer (wireless user), who pays to obtain the channel access.

Yet, it is important to point out that transactions take place in secondary markets, where spectrum is leased in a seller-defined geographic region. Unlike physical commodities that can be traded anywhere in the world, spectrum is a local resource and can only be leased to local users who are within the license region of this channel. In this sense, a local market is formed between the license holder and all these local users.

To facilitate such local markets, the FCC has advocated to establish an Internet database to store the information of the currently vacant spectrum as well as its geographic area [2]. An unlicensed user can simply query the database to obtain the list of channels that can be used at the user's location. This database-driven technique enables practical spectrum markets, such as SpecEx [3], to take full advantage of the spectrum locality and to provide flexible selling options to attract participation. One of them is license partitioning [3], in which a license holder can divide the entire geographic area in which it holds its spectrum license, called its license area, into several regions and sell any of the pieces to wireless users, as shown in Fig. 1.

Unfortunately, market locality, as an inherent characteristic of spectrum resources in practice, is seldom mentioned in the literature. Most existing spectrum auctions [4], [5], [6], [7], [8], [9] are designed in the sense of the global market—all channels are accessible to all users, no matter where they are. Furthermore, no license partitioning is supported: All channels have to be traded as a whole in the entire license area. These assumptions not only are impractical, but will also seriously degrade the flexibility of selling options, leaving license holders unable or unwilling to join the market. It is typical for channels to be only available to wireless users in some seller-defined geographic regions, rather than the entire license area of a license holder. For example, in Fig. 1, a channel may be vacant in region A, yet utilized by its license holder in region B and C. Without the support of license partitioning (as is the case in global market), a license holder has to decide if it is able to make channels vacant in its entire license area, and channels that are available only in part of the regions are not ready for sale.
Fig. 1. A license holder partitions its entire license area into three regions, A, B, and C. It can sell any of the pieces in the local spectrum market.

To bridge such a gap between the existing literature and practical limitations of geographic spectrum locality, in this paper, we present District, a set of new spectrum double auctions that are specifically designed for local spectrum markets. With District, a license holder can freely partition its entire license area and either sell or reserve spectrum in local markets, based on their own requirements. Moreover, District allows the same channel to be shared by multiple wireless users if no interference occurs.

We believe that it is crucial for District to maintain basic properties of economic robustness (truthfulness in particular). As a matter of fact, introducing the notion of local markets imposes nontrivial challenges when economic robustness is to be maintained. Most existing spectrum double auctions [5], [7], [8] are based on McAfee’s design [10], which is for the global market only. Their direct extensions, as shown in Section 3.6, are either not feasible or leading to fairly inefficient outcomes. To maintain economic robustness, District is designed to work effectively in cases with and without a priori information about bid distributions. In the former case, District extends Myerson’s virtual valuations [11] to double auctions and designs a market with a discriminatory pricing policy—different auction winners might face different charges or payments. In the latter case, District sets a uniform pricing mechanism to charge all winners uniformly. Both mechanisms are proved to be computationally efficient and economically robust. With minor modifications, the design is also applicable to a profit-driven auctioneer aiming to maximize the auction revenue. Extensive simulation studies show that District substantially improves spectrum utilization with local markets, and is scalable to large networks.

2 RELATED WORK

Auction serves as an efficient mechanism to price and distribute scarce resources in a market. Krishna [12] offers an excellent treatment of the general auction theory.

When it comes to spectrum auctions, pioneering works include [13], [14], [15], and [16]. All of them focus on the primary markets, where primary users bid to obtain long-term spectrum rights from the government.

Recently, spectrum auction has received considerable attention in dynamic spectrum access. Some early works in this field include transmit power auctions [17] and spectrum band auctions [18], [19], [20]. Gandhi et al. [19] aim to generate maximum revenue for sellers and employ linear programming to model interference constraints. A similar objective is also adopted in [20], where a greedy graph coloring algorithm is used to maximize the revenue in cellular network. Buddhikot and Ryan [18] also formulate the channel management problem in homogeneous CDMA networks and propose a spectrum allocation algorithm with linear programming. However, none of these works discusses the strategic behaviors of participants, and the claimed performances are only achieved when all participants truthfully reveal their private information.

The strategic issues of participants are first addressed in [4], in which two truthful spectrum auctions are designed to facilitate as many transactions as possible. Jia et al. [6] further present a truthful design to maximize the auction revenue. Also, [21] allows sellers to have reservation prices on their idle channels. All these designs are single-sided auctions where only one side, either spectrum buyers or spectrum sellers, has bidding strategies. When the strategic issues extend to both sides, [5] proposes the first spectrum double auction that is truthful. Based on it, [7] designs a double auction for spectrum secondary markets, where participants are all secondary users.

Besides truthfulness, recent works also consider other factors in spectrum auction designs. For example, [8] includes the time domain into strategy considerations and proposes a truthful online auction, while [9] investigates the fairness issue in spectrum allocations.

However, all works above discuss auction designs in the sense of global markets, where all channels to be auctioned off are globally accessible to all spectrum buyers, no matter where they are. Such ignorance of the geographic locality of spectrum resources made them incapable to accommodate the demand of the recent push of database-driven spectrum markets [2], [3], in which channels are traded to local users within the seller-defined license areas. A more detailed technical discussion is given in Section 3.6.

3 BASIC SYSTEM MODEL

This section presents the basic system model where a non-profitable auctioneer runs a spectrum double auction for unit-supply sellers and unit-demand buyers. By “unit” we mean each participant either sells or requests a single channel in the local market. In Section 6, we further discuss how this model can be extended to scenarios where 1) a buyer can request more than one channel, and 2) the auctioneer is profit-driven aiming to maximize its revenue.

3.1 General Settings

To accommodate spectrum requests that arise dynamically over time, an auctioneer (e.g., the spectrum database administrator) carries out a sequence of double auctions periodically, one in each time slot. Participants, both license holders (sellers) and wireless users (buyers), join the auction asynchronously and submit requests anytime they wish. Yet, their submissions may not be processed immediately until the next auction is conducted, at which time the auction results are computed. Due to the temporal dynamics of spectrum usage, spectrum leases are necessarily ephemeral. A mature business model to facilitate such short-term leases is to adopt an agreed-upon lease cycle, in which a buyer will hold a seller’s spectrum license after the transaction is made. Participants may join subsequent auctions to lease channels for additional cycles. Throughout
this paper, we focus on auction designs in one time slot, as they are the essential building blocks of the spectrum market and are independent of the value of the time slot or the lease cycle.

In each auction round, every seller has one channel for sale in an indicated license area (a.k.a., local market) in which the channel is vacant (e.g., region A in Fig. 1). Each seller reports the channel, the associated local market (i.e., the available geographic region), and an ask to the auctioneer. Each buyer, on the other hand, requests to buy one channel at some geographic location by submitting a bid and its location to the auctioneer. All bids and asks are submitted in a sealed manner—no one has access to any information about the others’ submissions. After collecting all these submissions, the auctioneer computes the best set of channel transactions to clear the market.

The main challenge is to establish proper payoff schemes and to optimally match buyers and sellers, with the constraints that all channel transactions must be made within local markets, and that no interfering buyers are assigned to the same seller. Fig. 2 illustrates an example of such a double auction with multiple spectrum sellers and buyers in different local markets. Note that the areas of local markets are drawn as circles only for illustration. In fact, they can have arbitrary shapes and may not even be contiguous in general. We finally assume there are $M$ participating license holders and $N$ wireless users.

### 3.2 Modeling Channel Transactions within Local Markets

Channels should be assigned without introducing interference. We use a conflict graph $G=(V,E)$ to represent interference relations among buyers, where $V$ is the collection of buyers and $E$ is the collection of edges, such that two buyers share an edge if they are in conflict with each other and cannot use the same channel simultaneously. In our example shown in Fig. 2, seven conflicting pairs of buyers are illustrated by dotted lines.

We say seller $m$ and buyer $n$ are tradable if $n$ is within $m$’s local market so that it can trade with $m$. We use $C_n$ to denote the set of tradable sellers of a buyer $n$. For example, in Fig. 2, $B1$’s tradable sellers are $S1$ and $S5$, i.e., $C_{B1} = \{S1, S5\}$.

Now, the network scenario can be represented by a conflict graph $G$ as well as all buyers’ tradable sellers $\{C_n\}$. As an example, Fig. 3 illustrates such a representation of the scenario depicted in Fig. 2, with $C_B$ labeled next to $B_i$ ($i = 1, \ldots, 7$).

A channel assignment scheme is feasible if all transactions are made between tradable sellers and buyers, and no two buyers sharing an edge in $G$ are assigned to the same seller. A feasible assignment can be equivalently converted to a feasible graph coloring scheme by treating tradable sellers $C_n$ as the available colors that can be used to color node $n$ in $G$. In this sense, a buyer $n$ is assigned to a tradable seller $m$ if node $n$ in $G$ is colored by $m \in C_n$, and vice versa. As an example, one feasible channel assignment of the scenario depicted in Fig. 2 is shown in Fig. 3, with the assigned spectrum underlined in the text. It can be seen that the assignment is exactly a feasible graph coloring scheme. We say a buyer $n$ (a seller $m$) is a winner if node $n$ is colored (color $m$ is used) in $G$; otherwise, it is a loser. For notational convenience, we integrate the available colors of nodes into the conflict graph and denote it by $G = (V, E, C)$, where $C = \{C_n | n \in V\}$.

### 3.3 Spectrum Double Auction

With the knowledge of $G$, the auctioneer collects asks (bids) from the sellers (buyers). Denote by $a_m$ and $b_n$ the ask and bid submitted by the seller $m$ and buyer $n$, respectively. Every seller $m$ has a true ask $a_{n,m}$, a price that it believes its channel is worth. Every buyer $n$ also has a true bid $b_{n,i}$, a price quantifying its economic benefit of getting a channel. The value of $a_{n,m}$ ($b_{n,i}$) is the private information of the seller $m$ (buyer $n$), and is unknown to anyone else, including the auctioneer. Note that the seller $m$ may submit a different ask from its true ask (i.e., $a_m \neq a_{n,m}$), as long as it believes that this is more beneficial. Similar misreporting strategy may also be adopted by any buyer $n$ (i.e., $b_n \neq b_{n,i}$).

After collecting all asks $a = (a_1, \ldots, a_M)$ and bids $b = (b_1, \ldots, b_N)$, the auctioneer clears the market by computing the channel assignment and winner payoffs. The assignment is represented by a coloring scheme of the conflict graph $G$, as mentioned before. Every winning seller $m$ is paid $p_m$ by the auctioneer for leasing a channel, while every winning buyer $n$ is charged $c_n$ by the auctioneer. Therefore, the payoffs consist of both the payments $p = (p_1, \ldots, p_M)$ to sellers and the charges $c = (c_1, \ldots, c_N)$ to buyers. Then, for each winning pair, the utility of seller $m$ is
Definition 2 (Budget balance). - An auction mechanism is said to be individually rational if every participant’s utility is nonnegative, i.e., \( u^a_m(r_n, b_n) \geq 0 \) for all \( n = 1, \ldots, N \), and \( u^a_m(r_n, b_n) = 0 \) for all \( m = 1, \ldots, M \).

By joining the auction with individual rationality, all participants are guaranteed to be benefited. This property is critical in attracting user participation.

Also, to make the auction self-sustained without any external subsidies, the generated revenue is required to be nonnegative.

Definition 2 (Budget balance). - An auction mechanism is said to achieve the ex post budget balance if its revenue \( \gamma \) is always nonnegative, i.e., \( \gamma = \sum_{n=1}^{N} c_n - \sum_{m=1}^{M} p_m \geq 0 \). Furthermore, if the revenue is nonnegative in expectation, the auction is said to be ex ante budget balanced. That is,

\[
E[\gamma] = E[\sum_{n=1}^{N} c_n - \sum_{m=1}^{M} p_m] \geq 0.
\]

Moreover, the designed spectrum double auction is preferred to be truthful.

Definition 3 (Truthfulness). - An auction is said to be truthful if no participant can expect a higher utility by misreporting its true submission. That is, for every seller \( m \) (respectively, buyer \( n \)), \( u^a_m(a^*_m) \geq u^a_m(a_m) \) (respectively, \( u^a_n(b^*_n) \geq u^a_n(b_n) \)) for all \( a_m \) (respectively, \( b_n \)).

As an important economic property, truthfulness brings the following benefits. First, a truthful mechanism helps the auctioneer to gather more accurate market information. We show later in Section 5 that utilizing such information enhances the auction performance. Second, truthfulness simplifies all participants’ strategies and ensures the basic market fairness. Since no one has the incentive to cheat at its submission, the auction result will not be manipulated by any individual participant. Finally, by the Revelation Principle [11], as long as the revenue is of interest, restricting discussions to truthful auctions does not lose generality.

Following the convention adopted in the existing literature, we say an auction is economically robust if it is individually rational, budget balanced (either ex post or ex ante), and truthful [5], [8].

3.5 Problem Definition

In the basic model, we consider a nonprofit auctioneer whose objective is to improve channel utilization by facilitating as many wireless users as possible to access idle channels. For an auction mechanism \( M \), we define the auction efficiency as the proportion of winning buyers, i.e., \( \eta_M = N_w / N \), where \( N_w \) is the number of winning buyers.

With input \( G = (V, E, C) \), asks \( a \) and bids \( b \), an auction mechanism \( M \) outputs payments \( p \), charges \( c \), and a colored graph \( G \) representing the channel assignment result. Ideally, we would like to find an economically robust auction mechanism that also maximizes the auction efficiency. However, the impossibility theorem [23] dictates that the maximal auction efficiency is incompatible with economic robustness. In this work, we view economic robustness as a hard constraint to ensure a well-behav- ing spectrum market. Hence, we are concerned with the following optimization problem:

\[
\max_{M} \eta_M
\]

s.t. \( M \) is economically robust. (1)

3.6 Challenges of Local Markets

Introducing the notion of local markets imposes nontrivial challenges when economic robustness is to be maintained. Simply extending existing spectrum double auctions, for example, [5], [7], [8], is either infeasible or leads to fairly inefficient outcomes. In [7] and [8], spectrum reuse is not considered, and a complete conflict graph is assumed. Neither of them is applicable to our system model.

For other auctions designed under a global market that consider spectrum reuse, say, TRUST in [5], it is possible to propose a simple extension for local markets when the traded license areas are of some special shape. For example, suppose all license areas are circular. As shown in Fig. 4, we can partition the entire geographic region into hexagonal cells, each with edge length \( R/2 \), where \( R \) is the minimum radius of all circular license areas. Then, it is always feasible for a buyer to trade with any seller whose license area is centered within the same cell of the buyer. If we limit our discussion to the buyers and sellers within one cell, then TRUST is applicable. Fig. 4 illustrates an example, where the extension is applied to the scenario depicted in Fig. 2.

However, such an extension is problematic. First, the traded license areas need to be of some special shape, which is usually not the case in practice. Second, buyers are only allowed to trade with sellers within the same cell. As a
result, many originally feasible transactions are blocked. For example, in Fig. 4, buyer $B_6$ cannot trade with seller $S_3$ since they are not covered by the same cell. However, in the original network scenario in Fig. 2, $S_3$ and $B_6$ are tradable. Similar trade-blocking phenomenon can also be observed among $B_5$, $B_7$, and $S_2$. As a result, $B_5$, $B_6$, and $B_7$ will be left unassigned, independent of their submissions. This intuitive example reveals that the auction efficiency of such simple extension is fairly low, especially when the cell is small. We later verify this problem by our numerical results, shown in Section 7.

The failure to simply extend auction mechanisms for global markets requires a new design tailored to local markets. The proposed District contains a set of auction mechanisms that are specifically designed to address unique challenges imposed by market locality. We present two alternative designs, District-D and District-U, for the cases with and without a priori knowledge on bid distributions, respectively. An auctioneer without any distribution knowledge can start with District-U for a moderate level of auction efficiency, and then switch to District-D to pursue a higher level of efficiency after collecting sufficient information about bid distributions.

4 District-U: Auction with Uniform Pricing

District-U adopts uniform pricing policies, such that all buyers (sellers) are charged (paid) exactly the same amount of money if they win, without a priori knowledge on bids or asks. The basic idea is trade reduction: nonprofitable trades among low-bid buyers and high-ask sellers are removed, which is critical in maintaining ex post budget balance. At a price, however, the auction efficiency is limited due to the reduction of feasible transaction pairs.

4.1 Preliminaries

Without loss of generality, we assume bids and asks are sorted, respectively, from the most competitive to the least competitive, i.e., $b_1 \geq b_2 \geq \cdots \geq b_N \geq 0$ and $0 \leq a_1 \leq a_2 \leq \cdots \leq a_M$. We also assume $M \geq N$. Otherwise, we add $N - M$ dummy sellers who ask $a_M$ for their channels but are not tradable to any buyers. Clearly, adding dummy sellers does not affect the auction result.

We introduce the following notations for convenience:

- $G_{m,n}$: A subgraph of $G = (V, E, C)$ composed of only the first $m$ colors (sellers) and the first $n$ nodes (buyers). That is, $G_{m,n} = (V', E', C')$ where $V' = \{1, \ldots, n\}$, $E' = \{e_{ij} \in E | i, j \in V'\}$, and $C' = \{C_1', \ldots, C_m'\}$ where $C_j' = C_j \cap \{1, \ldots, m\}$.
- $\text{GraphColoring}(G)$: A graph coloring algorithm returning a colored $G$. District-U accepts only a deterministic graph coloring algorithm without randomness introduced.

4.2 Mechanism Design

To achieve the trade reduction, District-U first decides how many buyers to admit, based on the submitted asks and bids. The auctioneer then computes the set of admitted sellers, and subsequently determines the transaction pairs and corresponding payoffs. The details are given in Algorithm 1.

Algorithm 1. District-U.
1. $N' = \arg \max_i \{a_{i+1} \leq b_{i+1}\}$
2. $M' = \arg \max_i \{a_i \leq b_{N'+1}\}$
3. $\text{GraphColoring}(G_{M',N'})$
4. Seller $m$ trades with buyer $n$ if node $n$ is colored by $m$.
5. for each winning buyer $n$ do
6. $c_n = b_{N'+1}$
7. end for
8. for each winning seller $m$ do
9. $p_m = b_{N'+1}$
10. end for
11. return transactions and payoffs ($e$ and $p$).

The key to understanding Algorithm 1 is to appreciate its trade reduction nature. As shown in Fig. 5, the algorithm first removes low-bid buyers, i.e., buyers $N'+1, \ldots, N$, based on the formula of line 1. It then uses the highest bid among the set of removed buyers (i.e., $b_{N'+1}$) as an admission threshold to determine how many sellers to admit: Those who ask higher than the threshold price are all rejected. After the trade reduction, each enrolled buyer bids at least as high as the ask of every admitted seller.

Note that even for the enrolled buyers and sellers, there is no official guarantee that they will eventually win out. The final spectrum assignment will be calculated by running a graph coloring algorithm, and prices are uniformly set for both winning buyers and winning sellers. We will see later that this uniform price applied to all buyers (sellers) is the least (highest) submission for them to win. Fig. 6 shows an example of applying Algorithm 1 to
the scenario depicted in Fig. 2, where we assume $N' = 6$ and $M' = 3$. We can see that participants $B7, S4, S5$ are directly removed by the auction.

### 4.3 Economic Properties

Intuitively, one might think that the economic properties of District-U should depend on the specific form of GraphColoring() adopted by Algorithm 1 in line 3. Surprisingly, the following analysis shows a general result saying that District-U is guaranteed to be economically robust, as long as the GraphColoring() is a deterministic algorithm. That is, with the same input $G$, algorithm GraphColoring() should always produce the same output.

To see this interesting result, we first prove the ex post budget balance directly. We then prove that District-U is bid monotonic with critical payoffs, which leads to truthfulness and individual rationality [22], [24].

**Proposition 1.** District-U is ex post budget balanced.

**Proof.** Denote by $x$ and $y$ the number of winning buyers and sellers, respectively. Since a seller’s channel could be spatially reused by multiple buyers, we have $x \geq y$. In this case, the auction revenue is always positive, i.e., $\gamma = \sum_{n=1}^{N} c_n - \sum_{m=1}^{M} p_m = (x - y) \cdot b_{N+1} \geq 0$. \(\blacksquare\)

Before we proceed to the formal proof of truthfulness and individual rationality, we first introduce two important concepts, the monotonicity and criticality. Both of them serve central roles in the proofs.

**Definition 4 (Monotonicity).** An auction mechanism is bid monotonic such that for every buyer $n$ (seller $m$), if by submitting $b_n$ ($a_m$) it wins, then by submitting $b'_n > b_n$ ($a'_m < a_m$) it also wins, given the others’ submissions remain unchanged.

The interpretation of Definition 4 is quite straightforward: a more competitive submission never hurts a participant’s chance to win. An important property associated with a bid monotonic auction is the unique existence of the critical submission defined below.

**Definition 5 (Criticality).** For winning buyer $n$ (seller $m$), we say $b'_n$ ($a'_m$) is critical if $n$ ($m$) wins by submitting $b_n > b'_n$ ($a_m < a'_m$) and loses by submitting $b_n < b'_n$ ($a_m > a'_m$), given the others’ submissions remain unchanged.

In other words, the critical bid (ask) is the minimum (maximum) submission for a buyer (seller) to win the auction, and is therefore a threshold submission in determining their auction results. Its value depends on other buyers’ bids (sellers’ asks) and how the winners are selected from the participants.

There is one important result in mechanism design regarding truthfulness and individual rationality, as described in Lemma 1.

**Lemma 1 ([22]).** A bid monotonic auction is truthful and individually rational if and only if it always charges critical bids from winning buyers and pays critical asks to winning sellers.

Now, we show that District-U is bid monotonic with critical payoffs, and is hence truthful and individually rational.

**Proposition 2.** District-U is bid monotonic.

**Proof.** Suppose buyer $n$ wins by bidding $b_n$. According to the admission rule stated in line 1, we have $b_n \geq a_n$ and $b_{n+1} \geq a_{n+1}$. Hence, by unilaterally raising the bid up to $b'_n$, buyer $n$ will be admitted again, as $b'_n > b_n \geq a_n$ and $b_{n+1} \geq a_{n+1}$. We see $G_{M',N'}$ remains unchanged. Since algorithm GraphColoring() is deterministic, with the same $G_{M',N'}$ as input, the output is always the same, which implies that $n$ wins again.

Similarly, consider a winning seller $m$ decreasing its ask unilaterally from $a_m$ to $a'_m$. Then, $a'_m < a_m \leq b_{N+1}$. That is, $m$ remains among the bottom $M'$ sellers, and will be admitted again. The rest of the proof is similar to the buyer’s case. \(\blacksquare\)

**Proposition 3.** In District-U, $b_{N+1}$ is the critical submission for all winning buyer $n$ and winning seller $m$.

**Proof.** For a winning buyer $n$, it will remain in the set of the top $N'$ buyers as long as it bids $b_n > b_{N+1}$, which ensures it to be admitted with the same $G_{M',N'}$. Then, $n$ keeps winning due to the unchanged graph coloring result.

On the other hand, suppose $n$ bids $b_n < b_{N+1}$ and is ranked in the $k$th place among all buyers, where $N' + 1 \leq k \leq N$. The rankings of bids now become $b_1 \geq \cdots \geq b_{n-1} \geq b_{n+1} \geq \cdots \geq b_k \geq b_{n+1} \geq b_{k+1} \geq \cdots \geq b_N$. Since $k + 1 > N' + 1$, by the definition of $N'$ (i.e., line 1 of Algorithm 1), we have $b_{k+1} < a_{k+1}$. This essentially indicates that buyer $n$ will not be admitted in the new rankings as its next bid $b_{k+1}$ is lower than the corresponding ask $a_{k+1}$ (refer to line 1 of Algorithm 1 for the admission rule).

We now consider the seller’s case. For a winning seller $m$, it will remain in the set of top $M'$ sellers as long as it asks $a_m < b_{N+1}$, which ensures it to be admitted and win (similar to the buyer’s case).

For the case where $m$ asks $a_m > b_{N+1}$, we see it must be removed because its ask exceeds the admission threshold (see Fig. 5 and line 2 of Algorithm 1). Therefore, $m$ must lose. \(\blacksquare\)

With Propositions 1, 2, and 3 and Lemma 1, we conclude the following:

**Theorem 1.** District-U is economically robust.

It is worth mentioning that the statement of Theorem 1 is independent of the underlying conflict graph. Therefore, a participant will always behave truthfully no matter how its belief on the conflict graph is.

Also note that the adopted graph coloring algorithm plays a key role in District-U as it not only affects the auction efficiency, but also dominates the computational complexity. Fortunately, since District-U accepts any deterministic coloring algorithms, those computationally efficient designs with good coloring performance could be directly applied. We later evaluate how the choice of the coloring affects the performance of District-U in Section 7.
In conclusion, District-II is designed as a suitable starting mechanism for the auctioneer without a priori information on bids or asks.

5 DISTRICT-D: AUCTION WITH DISCRIMINATORY PRICING

When bid and ask distributions are available, one can expect higher efficiency via District-D, an auction with discriminatory pricing policies (i.e., winners have different payoffs). We begin by using Myerson’s virtual valuations [11] to characterize the expected revenue of a truthful spectrum auction \( M \). We show that an economically robust \( M \) is equivalent to a graph coloring with a weighted sum nonnegative in expectation. We design District-D based on this result and show that the design is computationally efficient.

5.1 Preliminaries

Assume all buyers (sellers) bid (ask) independently, but possibly under different distributions. For every buyer \( n \), denote its bid distribution function as \( F^b_n(b_n) \) and the corresponding density function as \( f^b_n(b_n) \). We similarly define \( F^s_m(a_m) \) and \( f^s_m(a_m) \) for every seller \( m \). Since \( M \) is designed to be truthful, we do not discriminate the true bid (ask) and the submitted bid (ask) in the following context.

In [11], Myerson defines virtual valuations for buyers in a single-sided auction. We extend this idea to double auctions and apply it to our design. Formally, we define

\[
\psi_m(a_m) = a_m + \frac{F^s_m(a_m)}{f^s_m(a_m)}
\]

as the virtual valuation of seller \( m \) with ask \( a_m \), and

\[
\phi_n(b_n) = b_n - \frac{1 - F^b_n(b_n)}{f^b_n(b_n)}
\]

as the virtual valuation of buyer \( n \) with bid \( b_n \). We assume regular distributions [11], i.e., all \( \phi_n(\cdot) \) and \( \psi_m(\cdot) \) are increasing functions.¹

Let \( v = (a_1, \ldots, a_M, b_1, \ldots, b_N) \) be the vector of submissions. Denote by \( \gamma_M(v, G) \) the revenue of auction \( M \) with submissions \( v \) and the conflict graph \( G \) as the input. When there is no confusion, we simply write \( \gamma_M(v, G) \) as \( \gamma(v) \). The following lemma shows that the expected revenue can be fully characterized by the virtual valuations of all winners. The proof is similar to [11] and is given in the appendix.²

**Lemma 2.** Given a truthful \( M \) and \( G \), let \( x_n(v) = 1 \) if \( n \) wins, i.e., \( n \) is colored, and \( x_n(v) = 0 \) otherwise. Let \( y_m(v) \) be similarly defined for a seller \( m \). Then,

\[
E_v[\gamma(v)] = E_v \left[ \sum_{n=1}^{N} \phi_n(b_n) x_n(v) - \sum_{m=1}^{M} \psi_m(a_m) y_m(v) \right].
\]

Lemma 2 reveals an important fact—dealing with virtual valuations is equivalent to dealing with submitted bids (asks), in terms of the expected revenue.

Introducing \( \phi_n(\cdot) \) and \( \psi_m(\cdot) \) greatly simplifies the auction design problem. Suppose the conflict graph \( G \) is given. For a buyer \( n \) with bid \( b_n \), we assign \( \phi_n(b_n) \) as the node weight to node \( n \). For seller \( m \) with ask \( a_m \), we assign \( \psi_m(a_m) \) as the color weight to color \( m \). We define the weight of a colored graph \( G \) as

\[
W(G) = \sum_{n=1}^{N} \phi_n(b_n) \cdot x_n - \sum_{m=1}^{M} \psi_m(a_m) \cdot y_m.
\]

Here, \( x_n = 1 \) if \( n \) is colored, and \( x_n = 0 \) otherwise; \( y_m = 1 \) if \( m \) is used to color, and \( y_m = 0 \) otherwise. By Lemma 2, we have

\[
E_v[\gamma_M(v, G)] = E_v[W(G)].
\]

Therefore, achieving the ex ante budget balance is equivalent to maintaining the weighted sum nonnegative in expectation, i.e., \( E[W(G)] \geq 0 \).

Recall that the ultimate goal of our design is to pursue a high auction efficiency. We can now rewrite the objective problem (1) as follows:

\[
\max_M E[\eta_M] \\
\text{s.t. } E[W(G)] \geq 0,
\]

\( M \) is truthful and individually rational.

We see from the analysis above that the expected revenue of a truthful mechanism can be fully characterized by winners only, independent of their payoffs. In other words, for truthful spectrum auctions, the winner-determination algorithm (i.e., graph coloring) characterizes the pricing scheme, and serves a key role in District-D’s design.

5.2 Winner Determination

Based on our model, calculating the winning buyers and sellers is, in essence, to calculate a graph coloring scheme. From (7), we see that such a coloring scheme should color as many nodes as possible, while keeping the weighted sum nonnegative in expectation. Due to the intractability of this problem, our design is heuristic based.

In the proposed heuristic algorithm, at every iteration, we pick a feasible buyer-seller pair with the maximum marginal revenue measured by the virtual valuation. We then add the pair’s marginal revenue to the accumulated revenue. If a deficit (i.e., the resulted revenue is negative) occurs, then the pair is rejected and the algorithm terminates. Otherwise, the pair is accepted, and we proceed to the next iteration and repeat the above procedure.

For convenience, we introduce the following notations before presenting the formal algorithm in Algorithm 2:

- **S**—Set of all sellers \( S = \{1, \ldots, M\} \).
- **B**—Set of all buyers \( B = \{1, \ldots, N\} \).
- **T**—Round-by-round record of transactions already made by the winner-determination algorithm.
- **T^b**—Set of winning buyers corresponding to \( T \).
- **T^s**—Set of winning sellers corresponding to \( T \).
Algorithm 2. District-D Winner Determination
1. Initialization: $\gamma \leftarrow 0$, $T \leftarrow \emptyset$, and stop $\leftarrow$ false.
2. while stop $=$ false do
3. $\Delta_{m,n} \leftarrow \max_{T} \text{MarginalRev}(T)$
4. if $\gamma + \Delta_{m,n} \geq 0$ then
5. $\gamma \leftarrow \gamma + \Delta_{m,n}$
6. Add $(m,n)$ to $T$, color $G$ accordingly.
7. else
8. stop $\leftarrow$ true
9. end if
10. end while
11. return $T$

To better understand Algorithm 2, one can refer to Fig. 7 for an example.

Our winner-determination algorithm guarantees the budget balance, as stated by the following proposition.

**Proposition 4.** District-D is ex ante budget balanced.

**Proof.** Given conflict graph $G$, for all asks $a$ and all bids $b$, Algorithm 2 colors $G$ with a nonnegative weighted sum, i.e., $\gamma = W(G) \geq 0$. Hence, $E[W(G)] = E[\gamma_M(v,G)] \geq 0$, where the equality holds because of (6). Since this statement holds for every $G$, we conclude the proof. 

Also, our design has the potential to achieve truthfulness and individual rationality. The key to this point is to show that the design is bid monotonic. Once this is the case, the pricing mechanism will be an algorithm to calculate the critical submissions for all winners.

**Proposition 5.** Algorithm 2 is bid monotonic.

The formal proof of Proposition 5 is presented in the appendix, available in the online supplemental material. Intuitively, if buyer $n$ wins and trades with seller $m$ by bidding $b_n$, then raising the bid to $b'_n > b_n$ increases the marginal revenue generated by transaction $(m,n)$, i.e., $\Delta_{m,n}(T,a_m,b'_n) = \max_{v \in S,j \in B} \Delta_{i,j}(T,a_m,b'_n)$, as can be verified by referring to (8). Since $(m,n)$ is already the most profitable assignment when $n$ bids $b_n$, we see that in the new submission, $(m,n)$ remains the most profitable choice for the auctioneer and will be selected again. Therefore, $n$ wins by bidding higher than $b_n$. Similar argument also applies to winning seller $m$.

Now by Lemma 1, to achieve truthfulness and individual rationality, the pricing mechanism of District-D needs to calculate the critical submissions in a computationally efficient manner. We present the design in the next sections.

### 5.3 Buyer Pricing

By Definition 5, we see that the critical bid is the minimum submission required to win. The basic rationale is that to win the auction, there is no need to bid as high as possible. Instead, it suffices to win if one can do better than its competitors. Following this idea, we first remove buyer $n$ from bidding. We then conduct winner determination to obtain the winners list and see the winning competitors’ bids. Buyer $n$ can win as long as its bid is higher than the one submitted by the weakest competitor. The detailed procedure is given in Algorithm 3. It is worth mentioning that if the equation in line 4 (respectively,
Algorithm 3. District-D Pricing for a Winning Buyer \( n \).

1. Remove \( n \) and run Algorithm 2 to obtain the transaction (seller, buyer) list \( T = \{(i_1,j_1), \ldots, (i_k,j_k)\} \).
2. \( c_n \leftarrow -\infty \), and \( \gamma \leftarrow 0 \)
3. for \( l = 1 \) to \( k \) do
   4. \( b' = \min_{i \in S} b'_i \), where \( b'_i \) solves the following equation: \( \Delta_{i,n}(T_{l-1},a_i,b'_i) = \Delta_{i,j}(T_{l-1},a_i,b_j) \)
   5. \( c_n = \min \{c_n, b'_i\} \)
   6. \( \gamma \leftarrow \gamma + \Delta_{i,j}(T_{l-1},a_i,b_j) \)
   7. end for
8. \( b'^{k+1} = \min_{i \in S} b^{k+1}_i \), where \( b^{k+1}_i \) solves the following equation: \( \Delta_{i,n}(T_{k},a_i,b^{k+1}_i) = -\gamma \)
9. \( c_n \leftarrow \min \{c_n, b^{k+1}\} \)
10. return \( c_n \)

Let \( T \) be the transaction list generated by the winner-determination algorithm (Algorithm 2). Let \( T_i \) be the first \( l \) transactions in \( T \), i.e., \( T_i = \{(i_1,j_1), \ldots, (i_l,j_l)\} \), where \( (i_l,j_l) \) is the \( l \)th transaction made by Algorithm 2 between seller \( i_l \) and buyer \( j_l \). Denote by \( T^*_i \) the winning sellers associated with \( T_i \).

Proposition 6. For every winning buyer \( n \), Algorithm 3 returns its critical bid \( c_n \).

Proof. We first prove that \( n \) wins by bidding higher than \( c_n \), i.e., \( b_n > c_n \). It suffices to consider two cases:

Case 1. \( c_n \) is finalized in the first \( k \) loops, i.e., \( c_n = b' = b'_i \), for some \( i \leq k \) and \( i \in S \). For \( n \) bidding \( b_n > c_n = b'_i \), the worst case is that it loses in the first \( l - 1 \) rounds. But in the \( l \)th round, \( \Delta_{i,n}(T_{l-1},a_i,b_i) > \Delta_{i,n}(T_{l-1},a_i,b'_i) = \Delta_{i,n}(T_{l-1},a_i,b_j) \), where the first inequality holds due to the increase of \( \phi^*(\cdot) \) and (8). This implies that making the transaction \((i, n)\) will generate more marginal revenue than selecting \((i_l, j_l)\). Since \((i_l, j_l)\) is already the most profitable transaction when \( n \) is absent, we conclude that \((i, n)\) maximizes the marginal revenue when \( n \) joins the auction. Therefore, \( n \) wins by being selected to trade with seller \( i \).

Case 2. \( c_n = b^{k+1} = b^{k+1}_i \) for some \( i \in S \). For \( n \) bidding \( b_n > c_n = b^{k+1}_i \), the worst case is that it loses in the first \( k \) rounds. However, \( n \) can still trade with seller \( i \) after the first \( k \) rounds, with the marginal revenue \( \Delta_{i,n}(T_k,a_i,b_n) > \Delta_{i,n}(T_k,a_i,b^{k+1}_i) = -\gamma \). Therefore, adding the transaction pair \((i, n)\) to the auction results makes the total revenue remain positive, i.e., \( \Delta_{i,n}(T_k,a_i,b_n) + \gamma > 0 \). Based on the winner determination algorithm (Algorithm 2), \( n \) wins and trades with seller \( i \).

Next, if \( n \) bids less than \( c_n \) (i.e., \( b_n < c_n \)), then it loses in the first \( k \) rounds. For any \( l = 1, \ldots, k \) and any seller \( i \in S \), we have

\[
\Delta_{i,n}(T_{l-1},a_i,b_i) \leq \Delta_{i,n}(T_{l-1},a_i,b'_i) \\
= \Delta_{i,j}(T_{l-1},a_i,b_j).
\]

Here, the equality holds because of line 4. This essentially indicates that buyer \( n \) loses in the first \( k \) rounds as any transactions involving \( n \) does not generate the optimal marginal revenue.

Moreover, even if \( b^{k+1} < 0 \) after \( k \) rounds, \( n \) loses and cannot trade with any seller \( i \in S \). Otherwise, the total revenue would become negative: \( \Delta_{i,n}(T_k,a_i,b^{k+1}) + \gamma \leq \Delta_{i,n}(T_k,a_i,b^{k+1}_i) + \gamma = 0 \).

5.4 Seller Pricing

For sellers, the analysis on pricing is similar to buyers’. Seller \( m \) only needs to ask for lower than its competitors to win the auction. We first remove \( m \) and run the winner determination to see its winning competitors’ asks. Seller \( m \) can win by asking for lower than these competitors. The detailed design is given in Algorithm 4. Note that if the equation in line 4 (respectively, line 8) has no solution, then the value of \( a^{(i)}_j \) (respectively, \( a^{(k+1)}_j \)) is set to be \(-\infty\).

Algorithm 4. District-D Pricing for a Winning Seller \( m \).

1. Remove \( m \) and run Algorithm 2 to obtain the transaction (seller, buyer) list \( T = \{(i_1,j_1), \ldots, (i_k,j_k)\} \).
2. \( p_m \leftarrow -\infty \), and \( \gamma \leftarrow 0 \)
3. for \( l = 1 \) to \( k \) do
   4. \( a^l = \min_{j \in S} a^l_{ij} \), where \( a^l_{ij} \) solves the following equation: \( \Delta_{m,n}(T_{l-1},a^l_{ij},b_j) = \Delta_{i,j}(T_{l-1},a_i,b_j) \)
   5. \( p_m \leftarrow \max \{p_m, a^l\} \)
   6. \( \gamma \leftarrow \gamma + \Delta_{i,j}(T_{l-1},a_i,b_j) \)
   7. end for
8. \( a^{k+1} = \min_{j \in S} a^{k+1}_{ij} \), where \( a^{k+1}_{ij} \) solves the following equation: \( \Delta_{m,n}(T_k,a^{k+1}_{ij},b_j) = -\gamma \)
9. \( p_m \leftarrow \max \{p_m, a^{k+1}\} \)
10. return \( p_m \)

Proposition 7. For every winning seller \( m \), Algorithm 4 returns its critical ask \( p_m \).

The proof of Proposition 7 is similar to the proof of Proposition 6 and is given in the appendix, available in the online supplemental material.

5.5 Economic Properties and Computational Efficiency

From Propositions 5, 6, and 7, we see that District-D is bid monotonic and generates critical submissions. Therefore, by Lemma 1, it is truthful and individually rational. Noting that District-D is also ex ante budget balanced by Proposition 4, we conclude that:

Theorem 2. District-D is economically robust.

We now analyze the time complexity of District-D. In Algorithm 2, one transaction is made in each round of the loop, and the loop runs at most \( N \) rounds for \( N \) transactions. The complexity of each round is dominated by \( \text{MaxMarginalRev}(\cdot) \), which takes \( O(M^2N) \) time to finalize.\(^3\) We, hence, need \( O(M^2N^2) \) time for Algorithm 2.

\(^3\)A simple implementation is to go through all tradable transactions without conflicting with previously made trades, and select the one with the maximum marginal revenue. There are at most \( MN \) such transactions, each requiring at most \( M \) comparisons to clear the conflict relation. We hence need \( O(M^2N) \) time for \( \text{MaxMarginalRev}(\cdot) \).
Note that Algorithm 2 also dominates the complexity of Algorithms 3 and 4, where the former runs $N$ times while the latter runs $M$ times to calculate prices for all winners. We conclude that District-D completes within $O(M^2 N^3 + M^3 N^2)$ time.

6 Extensions

For now, all our discussions are based on a basic model where each buyer requests only a single channel and the auctioneer is nonprofit—it’s ultimate goal is to maximize the auction efficiency, not the revenue. In this section, we show that with minor modifications, previously developed techniques also apply to more general cases, in which 1) buyers can bid for multiple channels, and 2) the auctioneer aims to maximize the auction revenue.

Note that a more general model should also include multisupply sellers selling different channels idle in different geographic regions, at different prices. Each seller jointly optimizes the ask prices of different channels to maximize the overall utility. Since every seller has high-dimensional strategies, the problem is essentially a combinatorial mechanism design problem [22]. We are not aware of any revenue characterization result reported in this setting. Without this result, it would be very difficult to ensure budget balance, as there is no way to characterize the auction revenue. For these reasons, we leave this general scenario as an open problem for future investigation.

6.1 Multidemand Buyers

6.1.1 Models

Suppose every buyer $n$ requests $r_n$ channels at its location and has a true bid $b_n^t$ for accessing each one. Buyer $n$ reports $r_n$ and submits a bid $b_n$ to maximize its utility. No partial fulfillment is accepted: A buyer is either rejected or has all requested $r_n$ channels being allocated. We define buyer $n$’s utility $u_n^b(b_n)$ as follows:

$$u_n^b(b_n) = \begin{cases} r_n(b_n^t - c_n), & \text{if } n \text{ wins;} \\ 0, & \text{otherwise}, \end{cases}$$

where $c_n$ is the charged per-channel price. A buyer’s problem is to find an optimal submission to maximize its utility, i.e., $\max_{b_n} u_n^b(b_n)$. We note that this multidemand buyer model has also been adopted in [4], [21]. The seller model is the same as in Section 3.

The auctioneer’s problem is to design a spectrum double auction to maximize the auction efficiency $\eta_M$ while maintaining economic robustness. Here, $\eta_M$ is defined as the ratio of the fulfilled channel requests to the total requests, i.e., $\eta_M = \sum_{n \in W_b} r_n / \sum_n r_n$, with $W_b$ being the set of all winning buyers.

Note that the auctioneer has to ensure that all $r_n$ channels allocated to buyer $n$ are different, we add an interfering edge between every two subnodes of it. For convenience, denote by $G$ the extended conflict graph of the original $G$. A feasible channel assignment, therefore, corresponds to a coloring scheme on an extended conflict graph $G$.

6.1.2 Extending District-U

In Algorithm 1, after the trade reduction, instead of coloring the original conflict graph $G_{M,N}$ (see line 3), we color the extended graph $G_{M,N}$, and buyer $n$ wins if all its $r_n$ subnodes are colored. The pricing scheme remains the same as the original one: Every winning seller is paid $b_{n+1}$ and every winning buyer is charged $b_{n+1}$ for each traded channel.

It is easy to verify that both the claims and proofs of Propositions 1, 2, and 3 trivially apply to the above extensions, leading to the properties of truthfulness, bid monotonicity, and critical submissions. Therefore, we have the following theorem.

Theorem 3. The extended District-U is economically robust.

6.1.3 Extending District-D

Since a winning buyer $n$ now trades with $r_n$ sellers, by Lemma 2, in expectation, accommodating its channel requests generates $r_n \phi_n(b_n)$ revenue for the auctioneer. In fact, we see that Lemma 2 can be trivially extended to characterize the expected auction revenue as

$$E_v[\gamma(v)] = E_v \left[ \sum_{n=1}^{N} r_n \phi_n(b_n) x_n(v) - \sum_{m=1}^{M} \psi_m(a_m) y_m(v) \right].$$

Therefore, being budget balanced is equivalent to maintaining the RHS of (10) nonnegative.

Some notations are introduced before presenting the formal designs. Denote by $(S,n)$ the transactions made between buyer $n$ and a set of sellers $S = \{s_1, \ldots, s_m\}$. Let $S$, $B$, $T$, $T^*$, $T^s$, $T^t$, and $T^i$ be similarly defined as they are in Sections 5.2 and 5.3. Denote by $\Delta_{S,n}(T)$, the marginal revenue generated by adding transactions $(S,n)$ to the existing transaction list $T$. By (10), when adding $(S,n)$ to $T$ is feasible, we have

$$\Delta_{S,n}(T) = r_n \phi_n(b_n) - \sum_{i=1}^{r_n} \psi_i(a_i) I_{s_i \notin T^*}.$$  

Otherwise, $\Delta_{S,n}(T) = -\infty$. Sometimes, $\Delta_{S,n}(T)$ is also written as $\Delta_{S,n}(T,a_n)(\Delta_{S,n}(T,b_n))$ if seller $m$’s ask (buyer $n$’s bid) is of interest. Finally, let $\text{MaxMarginalRev}(T)$ return the feasible transactions $(S,n)$ with the maximum marginal revenue, given the already decided transaction list $T$, i.e., $\text{MaxMarginalRev}(T) = \max_{(S,n) \in B, S \subseteq S, |S|=r_n} \Delta_{S,n}(T)$.

Winner determination. The extended winner determination algorithm follows the same logic flow of Algorithm 2, with $\Delta_{m,n}$ and $(m,n)$ being replaced by $\Delta_{S,n}$ and $(S,n)$, respectively. Intuitively, the auctioneer greedily adds transactions $(S,n)$ with the maximum marginal revenue to its output, until the accumulated revenue is no longer positive, at which time it stops.

Buyer pricing. With the same design idea of Algorithm 3, we extend the buyer pricing to Algorithm 5. Note that in
Algorithm 5. Extended District-D Pricing for a Winning Buyer $n$.
1. Remove $n$ and run Algorithm 2 to obtain the transaction list $T = \{(S_1, j_1), \ldots, (S_k, j_k)\}$.
2. $c_n \leftarrow -\infty$, and $\gamma \leftarrow 0$
3. for $l = 1$ to $k$ do
4. \hspace{1em} $b^l \leftarrow \min_{S \in \mathbb{S}, \lvert S \rvert = r_s} b^l_{(S)}$, where $b^l_{(S)}$ solves the equation $\Delta_{S,n}(T_{l-1}, b^l_{(S)}) = \Delta_{S,j}(T_{l-1}, b_j)$.
5. $c_n \leftarrow \min\{c_n, b^l\}$
6. $\gamma \leftarrow \gamma + \Delta_{S,j}(T_{l-1}, b_j)$
7. end for
8. $b^{k+1} \leftarrow \min_{S \in \mathbb{S}, \lvert S \rvert = r_s} b^{k+1}_{(S)}$, where $b^{k+1}_{(S)}$ solves the equation $\Delta_{S,n}(T_k, b^{k+1}_{(S)}) = -\gamma$
9. $c_n \leftarrow \min\{c_n, b^{k+1}\}$
10. return $c_n$

Seller pricing. Algorithm 6 presents the design of the extended seller pricing. If the equation in line 4 (respectively, line 8) has no solution, we simply set $a^l_{(j)} = -\infty$ (respectively, $a^{k+1}_{(j)} = -\infty$).

Algorithm 6. Extended District-D Pricing for a Winning Seller $m$.
1. Remove $m$ and run Algorithm 2 to obtain the transaction list $T = \{(S_1, j_1), \ldots, (S_k, j_k)\}$.
2. $p_m \leftarrow -\infty$, and $\gamma \leftarrow 0$
3. for $l = 1$ to $k$ do
4. \hspace{1em} $a^l \leftarrow \max_{j \in \mathbb{U}} a^l_{(j)}$, where $a^l_{(j)}$ solves the equation $\max_{S \in \mathbb{S}, \lvert S \rvert = r_m} \{\Delta_{S,j}(T_{l-1}, a^l_{(j)})\} = \Delta_{S,j}(T_{l-1})$
5. $p_m \leftarrow \max\{p_m, a^l\}$
6. $\gamma \leftarrow \gamma + \Delta_{S,j}(T_{l-1})$
7. end for
8. $a^{k+1} \leftarrow \max_{j \in \mathbb{U}} a^{k+1}_{(j)}$, where $a^{k+1}_{(j)}$ solves the equation $\max_{S \in \mathbb{S}, \lvert S \rvert = r_m} \{\Delta_{S,j}(T_k, a^{k+1}_{(j)})\} = -\gamma$
9. $p_m \leftarrow \max\{p_m, a^{k+1}\}$
10. return $p_m$

Following the same analyses of Propositions 4 and 5, we can easily show that the winner determination is ex ante budget balanced and bid monotonic, while both the buyer and seller pricing return critical submissions. Therefore, we have the following theorem. The proofs are given in the appendix, available in the online supplemental material.

Theorem 4. The extended District-D is economically robust.

Note that all extensions above remain computationally efficient, taking $O(M^2 N^3 + M^2 N^2)$ time to complete. The detailed analysis is also given in the appendix, available in the online supplemental material.

6.2 Profit-Driven Auctioneer
Previous discussions assume a nonprofit auctioneer aiming to maximize the auction efficiency. Such an unselfish auctioneer could be governmental or public agencies such as the FCC in US. However, when the auction is operated by some profit-driven company, such as Spectrum Bridge, Inc., then the auctioneer’s objective is to maximize its own revenue. In this case, the auctioneer runs District-U first. After accumulating sufficient knowledge of bid distributions, it switches to District-D with the following minor modifications.

By Lemma 2, maximizing the expected auction revenue is equivalent to maximizing the RHS of (4), which is apparently an NP-Complete problem. We, hence, adopt a heuristic solution with greedy designs: Sequentially add a transaction $(m,n)$ with the maximum marginal revenue $\Delta_{m,n}$ to the auction output, until there is no profitable transaction to make (i.e., $\Delta_{m,n} < 0$). In particular, we change the “if” condition from $\gamma + \Delta_{m,n} \geq 0$ (see line 4 of Algorithm 2) to $\Delta_{m,n} \geq 0$. Some adjustments are also needed in the pricing design: We replace $-\gamma$ in line 8 of Algorithm 3 with 0. The same replacement is also taken in the same line of Algorithm 4.

With similar proofs to Propositions 5, 6, and 7, one can easily show that the extension above is bid monotonic and charges (pays) the winners their critical submissions. Therefore, we have the following conclusion.

Theorem 5. The extension of District-D for a profit-driven auctioneer is truthful and individually rational.

Finally, for multidemand buyers, the extensions presented in Section 6.1 are also applicable, and Theorem 5 extends to this scenario as well. The complete design is given in the appendix, available in the online supplemental material.

7 Simulation Results
We evaluate the performance of District with extensive simulations. Buyers are uniformly distributed in a $1 \times 1$ geographical region. Two buyers interfere with each other if their Euclidean distance is less than 0.1. Every seller indicates a license area to sell. The license area is assumed to be circular, with radius uniformly distributed between 0.2 and 0.5 and center uniformly located in the entire region. All bids and asks are normalized and follow the uniform distribution in the range of $[0, 1]$. Since District is truthful by design, these are also true bids and asks.

We evaluate the auction efficiency as the main performance metric of District, as our design is already proven to be economically robust. We also present numerical results for the generated revenue to study the relation between the auction efficiency and the revenue. All results presented in this section are based on the basic scenario where every participant trades only one channel in the local market. For the extended case where a buyer asks for multiple channels, we observe similar performance trends, which is reasonable because the extensions adopt similar design ideas to the original ones. Each result obtained below has been averaged over 10,000 runs.

7.1 District-U
We first investigate how the performance of District-U is affected by the choice of the deterministic graph coloring algorithm GraphColoring(). We compare three sequential coloring algorithms with different coloring orders, i.e., the fixed order where nodes with the least node ID is colored first, the least uncolored neighbors where a node with the least uncolored neighbors is colored first, and Brélaz’s
DSATUR where a node with the least available colors is colored first [25]. Fig. 8a illustrates the comparison results of their performances. Interestingly, we see that all three coloring algorithms achieve similar auction efficiency. This suggests that the performance of District-U is less dependent on the specific design of GraphColoring(). However, adopting a good coloring algorithm, such as DSATUR, does benefit the auction efficiency.

Since DSATUR slightly outperforms the other two coloring, our following simulations are carried out based on District-U with DSATUR. Figs. 8b and 8c further illustrate its mean and standard deviation of the auction efficiency, respectively, while Fig. 8d presents the mean revenue. We see that the mean auction efficiencies in all four experiments exhibit similar trend in their performance curves: Higher auction efficiency is achieved as more sellers join the auction, while the marginal enhancement is decreasing. On the other hand, when the channel supply is maintained unchanged, adding more buyers to the market degrades the auction efficiency, as the fixed amount of channel supply is insufficient to accommodate the increasing demand. When the channel supply and demand are of the same level (i.e., the number of buyers is equal to the number of sellers), the auction efficiency is around 50 percent.

It is worth mentioning that the maximum revenue does not always come with the optimal efficiency, as illustrated in Fig. 8d. Although enrolling more buyers increases the number of potential transaction pairs, it essentially lowers the admission threshold, making each transaction less profitable. However, high revenue is indeed extracted when the number of winners becomes a dominating factor.

7.2 District-D

Though District-U does not require a priori information and is ex post budget balanced, its efficiency is highly constrained due to its trade-reduction nature: many feasible trades are reduced to avoid a budget deficit. As shown in Fig. 8e, when the bid distribution knowledge is available, District-D can do better. As more sellers become available and the channel supplies increase, the auction efficiency can grow quickly, until the market is saturated with almost all buyers being served.

Interestingly, as shown in Fig. 8e, the more buyers join the market, the higher the mean auction efficiency is. We give an intuitive explanation here. The design rationale of District-D is to add as many trades as possible, with the constraint that the total revenue measured by virtual valuations is nonnegative in every iteration. However, not all transactions are profitable. These transactions are accepted only when the auctioneer has already accumulated sufficient revenue in the past trades: it can use its accumulation to compensate for the deficit caused by these unprofitable transactions while still maintaining budget balance. For a smaller market with fewer buyers, the accumulated revenue is limited and is insufficient to compensate for the deficit. As a result, the trade that is not profitable has to be dropped, leaving a relatively low auction efficiency compared with a market with more buyers. Fig. 8g validates this point of view—low efficiency usually comes with low revenue. As a conclusion, District-D is scalable to large networks.

Also, we depict the standard deviation of the auction efficiency in Fig. 8f. We see that the scenario containing more participants, either buyers or sellers, tends to have a smaller standard deviation. Moreover, the deviation is observed to be exponentially diminished with respect to the number of participants. One can view this result as an outcome of the Law of Large Numbers.

We next compare the performance of District-D and District-U with DSATUR by inspecting Figs. 8b, 8d, 8e, and 8g. By a significant margin, District-D outperforms
Finally, we evaluate the profit-driven District-D discussed in Section 6.2. We see in Fig. 9b that the design generates significantly higher revenues than both District-U and its efficiency-driven counterpart. At a price, however, the auction efficiency is much lower, as illustrated in Fig. 9a.

In conclusion, District-U serves as an appropriate transitional mechanism for the auctioneer to sustain the auction without external subsidies. It generates a moderate level of auction efficiency, and creates time for the auctioneer to collect the distribution information of bids and asks. When the distribution information is available, the auctioneer can switch to District-D to enjoy higher auction efficiency.

7.3 District versus a Simple Extension of TRUST

Section 6 discusses a scheme to directly extend an existing spectrum auction to make it compatible with local markets. A natural question is whether such a simple extension can provide acceptable performance. To study this, we extend TRUST\textsuperscript{4} as described in Section 6 and investigate its auction efficiency.

In Fig. 8h, we see that, with market sizes comparable to those experimented in District (i.e., fewer than 100 sellers and buyers), the auction efficiency of TRUST-extension is fairly low—generally less than 0.1—and grows slowly when channel supplies increase. Moreover, when more buyers are available in the auction, the market is saturated and the efficiency drops. This is in stark contrast to the scalability of District-D. The efficiency of TRUST-extension improves only when a very large amount of channel supplies are available in the market, but it is still severely limited when the number of buyers is small. By comparing Fig. 8h with both Figs. 8b and 8e, we conclude that District significantly outperforms the simple extension in Section 6.

8 Conclusion

In this paper, we present District, a set of new spectrum double auctions that incorporate market locality for practical spectrum markets, where sellers can freely partition their license areas to either sell or reserve, based on their own requirements. An auctioneer can start from District-U, a uniform pricing auction, to obtain moderate auction efficiency without any a priori information about bids. After accumulating sufficient knowledge of bid distributions, it can then switch to District-D, a discriminatory pricing auction, to pursue better auction efficiency. Our computationally efficient designs are proved to be economically robust and scalable to large networks. With minor modifications, our designs can also be applied by a profit-driven auctioneer to pursue the high auction revenue. To our knowledge, this is the first set of spectrum double auctions designed for local markets with these properties.

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References


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